

# Expansion of the Universe

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## Summary

This workshop contains several simple activities to do in which we are going to work with the key concepts of the expanding universe. In the first activity we build a spectroscope to observe spectra of gases. In the second, third, and fourth we experiment qualitatively with the expansion of a rubber band, a balloon, and a surface of points, respectively. In the fifth activity we work quantitatively with the expansion of a surface and even calculate the Hubble constant for this case. In the sixth activity we detect the microwave background radiation.

## Goals

- Understand the expansion of the universe.
- Understand that there is not a center of the universe.
- Understand Hubble's Law.
- Understand the meaning of the dark matter and simulate gravitational lens

## The Origin of the Universe

The theory of the origin of the universe that is most accepted today is known as the Big Bang, a huge explosion that began an expansion of space itself. There are not galaxies moving through space, but it is the space between them which expands, dragging the galaxies. For that reason, we may not speak of a center of the universe, as nobody can speak of a country that is in the center of the earth's surface.

The recession velocity of a galaxy is proportional to the distance it is from us. The constant that relates is called the Hubble constant. Hubble's law relates linearly the distance of a galaxy to the speed with which it moves away.

The first verification of the Big Bang came with the observation of redshifts in the spectra of galaxies, and the final proof to the Big Bang theory was the detection of the cosmic microwave background.

## Redshift

If at the laboratory we look with a spectroscope at the light coming from a hot gas, eg. hydrogen, we will see some colored lines that are typical of that gas at a determined wavelength. If we do the same with the light coming from a distant galaxy, we will see these

lines slightly displaced (figure 1). It's called redshift, because in most galaxies the lines are moving towards that color.

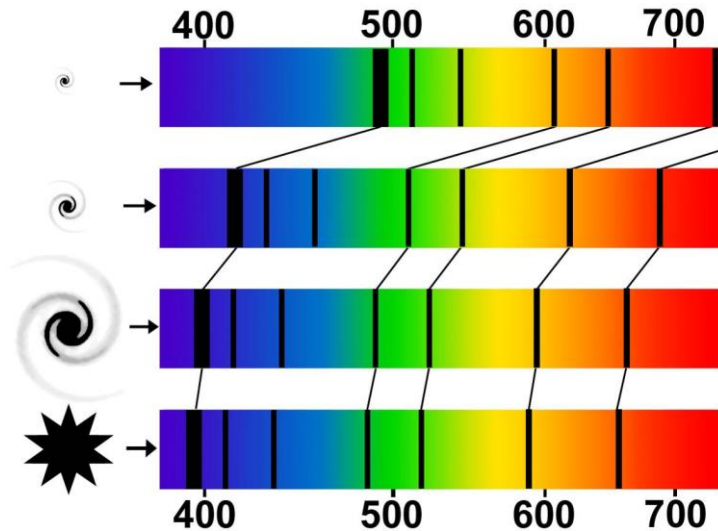


Fig.1: The farther the galaxy, the more the spectrum shifts towards red, which tells us that the galaxy is moving away from us faster.

The redshift of light is due to the flight of the galaxy away from us, similar to a locomotive whose whistle tone changes when it moves towards or away from us, and the larger the shift, the greater the speed.

Studying the spectrum of our local group galaxies, we find that the Large Magellanic Cloud is receding from us at 13 km/s, and the Small one is receding at about 30 km/s. Andromeda moves about 60 km/s towards us, while M 32 (one of its satellites) recedes at 21 km/s. In other words, nearby galaxies have small and irregular relative movements.

But if we look at the Virgo cluster, at an average distance of 50 million light years (ly) away, we see that all are receding from us at speeds between 1000 and 2000 km/s. And in the Coma of Berenice supercluster 300 million ly away, the speed rates are between 7000 and 8500 km/s. But looking in the opposite direction, we find that M 74 is receding from us at 800 km/s and M 77 at 1130 km/s. And if we look at galaxies more and more distant and faint, the recession velocity is even greater: NGC 375 moves at 6200 km/s, NGC 562 at 10,500 km/s, and NGC 326 at 14,500 km/s. All but the very close galaxies are moving away from us. Are they angry with us?

### Activity 1: Doppler effect

In the Doppler effect the wavelength of a sound varies when the source is moving. We experience it in the sound of motorbikes or cars in a race: the sound is different when approaching and moving away from us. Other familiar examples are a fire truck that passes by us, the whistle of a moving train, etc.

You can reproduce it spinning on a horizontal plane a buzzer, for example, an alarm clock. We place it into a cloth bag (figure 2a) and tie it with a string. When we spin it over our heads (figure 2b), we can hear it when it approaches the viewer:  $\lambda$  is shortened and the sound is

higher pitched. When it goes away from us, the  $\lambda$  is stretched and the sound is more bass, or lower pitched. The one in the center of rotation does not experience it.



Fig. 2a: Alarm clock, bag and string.



Fig. 2b: We revolve over our heads. Spectators off to one side notice the differences in the ringtone.

This is the Doppler effect due to displacement. But it is not the one that galaxies have with the expansion. The galaxies don't move through space, it is the space between them that expands.

### Activity 2: The "stretch" of the photons

The Universe, when it expands, "stretches" the photons in it. The longer the duration of the photon trip, the more stretching it undergoes.

You can make a model of that stretch with a semi-rigid cable, which is used in electrical installations of houses. Cut about one meter of cable, and bend it by hand making several cycles of a sinusoid, representing various waves (figure 3a).



Fig. 3a: Made waves with rigid cable.

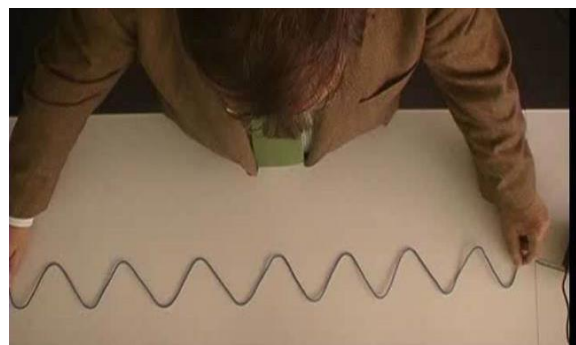


Fig. 3b: Same waves showing a longer wavelength.

Take the cable with both hands and stretch (figure 3b) and observe that the wavelength increases, as occurs in the radiation that comes from a galaxy. The parts farther away from us have had more time to stretch and moved further into the red ( $\lambda$  larger).

## Hubble's Law

It was Edwin Hubble (figure 4) who, leaning on these data, established in 1930 the law that bears his name: the more distant a galaxy is the faster it moves away from us. This indicates that the universe expands in all directions, so that all bodies that are in it are receding from each other. The movement away from us we see in all the galaxies does not mean that we are in the middle of them: an alien would look the same from anywhere in the universe, as happens in an explosion of fireworks: all light particles will be moved apart by the explosion of gunpowder.

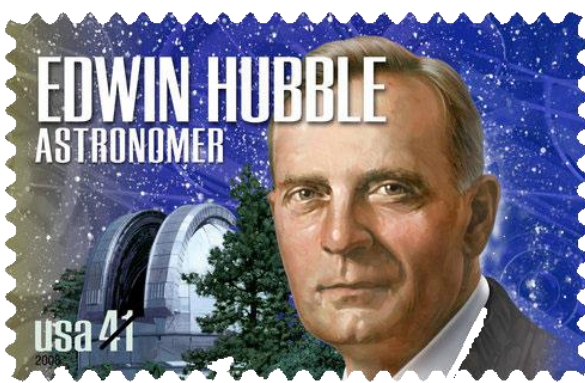


Fig. .4: Edwin Hubble.

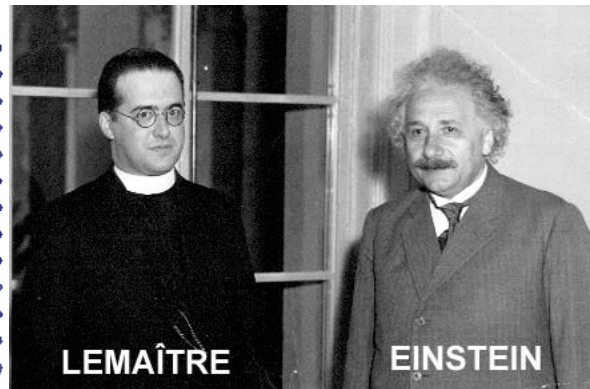


Fig. .5: George Lemaître and Albert Einstein.

However, the real model is not a galaxy moving through space, but it is the space between them which expands, dragging the galaxies.

If space expands in all directions, it means that if time were turned back, the matter should be focused on some initial moment where everything started.

That was how the Belgian priest and astronomer George Lemaître (figure 5) established the most widely accepted model of the universe today: there was an original big explosion, and in it we are still involved. In this expansion it is the space itself that expands. To understand this, imagine a rubber balloon with a series of points drawn on its surface, representing galaxies (figure 6). As it bulges, the elastic space between the speckles increases. Likewise, as time passes, the space will expand, and the contained substance itself is separating.

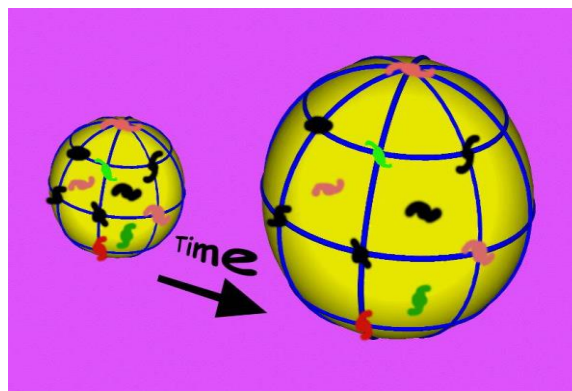


Fig. 6: As time passes, the space expand, and the material contained there in is separating from each other

Therefore, the recession velocity of a galaxy and its distance from us appears to be proportional. The constant that relates is called the Hubble constant. The Hubble law relates the distance of a galaxy with the speed with which it departs:

$$v=H \cdot d$$

One can roughly determine its value by knowing the speed and distance of some galaxies. The rate at which a galaxy is moving away is easy to measure accurately by the redshift, but measuring the distance, especially in the case of the more remote galaxies, is more difficult. Scientists do not agree on the value of the Hubble constant. Using one method or another, the emerging values generally range between 50 and 100 km/s per Megaparsec. The currently accepted value is approximately 70, indicating the age of the Universe to be 13,700 million years.

### Activity 3: The Universe in a rubber band

Edwin Hubble discovered that all galaxies are receding from us. The farther they are, the faster they do it. The so-called Hubble's Law states that the recession velocity of a galaxy relative to us is proportional to its distance. It is a logical consequence of the expanding universe. And although all galaxies are receding from us, it does not mean that we are the center of the universe.

With a marker, make a mark every centimeter on a rubber band. Each mark represents a galaxy (A, B, C, ...). Our galaxy will be the first one. Place the rubber next to the ruler (figure 7a), and allow our galaxy to coincide with the mark of 0 cm. The other galaxie A, B, C, ... coincide with the marks 1, 2, 3, 4 ... cm.

Stretch the rubber band (figure 7b) so that our galaxy remains at the 0 cm mark and that the following galaxy (A) be put on the 2 cm mark. The distance of this galaxy to our own has doubled. What happened to the distance between the other galaxies B, C, D and our own? Have they also doubled?



Fig. 7a: Rubber band without stretch.



Fig. 7b: Stretched rubber band.

Suppose that the time spent on the stretch of the rubber was 1 sec. Are the receding rates of the other galaxies all the same, or are some moving away faster than others? How does an inhabitant of the next "galaxy" see our galaxy and other galaxies? Do they also have all of them moving away?

### Activity 4: The universe in a balloon

Within the expanding universe, there is space between galaxies that expands. The galaxies themselves do not expand, nor do our houses. What is tightly bound by gravity does not increase in size.

There's a simple experiment that can demonstrate this. Just use a balloon and inflate it a little at first. Then paste a few pieces of cotton onto the surface with masking tape (coins also work). Then inflate the balloon until it is full. The pieces of cotton will be separated from each other (figures 8a and 8b). Some appear to go farther than others, but none become closer. It is a very simple model of the expanding universe.



Fig. 8a: Pieces of cotton glued to a slightly inflated balloon



Fig. 8b: The pieces of cotton move away when the balloon is more swollen

### Activity 5: Calculation of the Hubble constant

Hubble's Law says that the velocity  $v$  of a galaxy is proportional to the distance from us:  $v = H \cdot d$ . The constant  $H$  is called Hubble constant, and you can calculate it using distances and velocities of some galaxies. From the formula above:

$$H = \frac{v}{d}$$

The diagram of figure 12 shows space, represented by a blue grid of dashed lines, with us in the center and several blue galaxies at a distance from us. After some time, say 10 seconds, space has expanded and both the grid (in solid lines) and galaxies are represented in red.

Fill in table 1 beneath the drawing. In each row put in the data for each galaxy. For example, the coordinates are calculated with the blue squares (dashed lines) or red (solid lines) as galaxy A or A' respectively, and the distance  $d$  is obtained by measuring the length in centimeters with a ruler, starting at the center of our galaxy. The column data  $\Delta d$  must be obtained by subtracting the distance from A' and A. In the last column we must use the distance before expanding (eg A, not A') in the denominator.

Check that:

- The coordinates of each galaxy do not vary with the expansion (galaxies do not move through space).
- The value of  $H$  is fairly constant regardless of the galaxies.

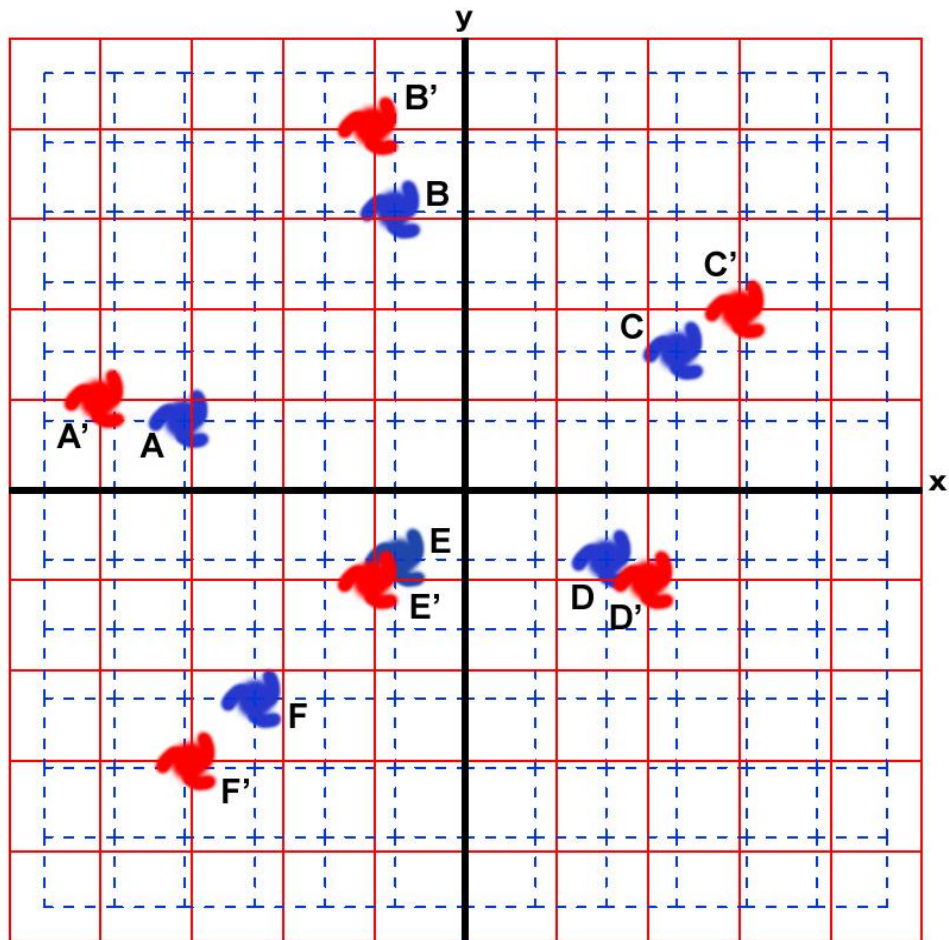


Fig 9: The grid of solid lines (red) is the same as the dashed one (blue) but expanded. The galaxies are attached to the grids.

Galaxia	Coordenadas <i>x,y</i>	<i>d</i> =distancia al origen	$\Delta d$	$v = \frac{\Delta d}{\Delta t}$	$H = \frac{v}{d}$
A	(-4, 1)				
A'	(-4, 1)				
B	(-1, 4)				
B'	(-1, 4)				
C	(3, 2)				
C'	(3, 2)				
D	(2, -1)				
D'	(2, -1)				
E	(-1, -1)				
E'	(-1, -1)				
F	(-3, -3)				
F'	(-3, -3)				

Table 1: with the coordinates written as an example

Galaxy	Coordinates <i>x,y</i>	<i>d=distance from origin</i>	$\Delta d$	$v = \frac{\Delta d}{\Delta t}$	$H = \frac{v}{d}$
A					
A'					
B					
B'					
C					
C'					
D					
D'					
E					
E'					
F					
F'					

Table 2: To be completed with data from figure 9.

## The Big Bang

Currently, the theory of the origin of the universe as a huge explosion is widely accepted in the scientific community, although there are those who doubt and feel that there are still details left unexplained. In 1994 the American magazine *Sky & Telescope* had a contest to rename it again. 12000 submissions were received, but none could unseat the one it already had: the Big Bang theory. The name was chosen as a disparaging one by the astronomer Fred Hoyle, who, with certain anti-religious bias, thought it seemed too consistent with the idea of a Creator.

With the observation of an expanding universe, it shows that in turning back the time there was a principle on which the explosion occurred, giving rise to space and time as we know it now. We may ask how it happened and why it happened. Science does not have an answer because it only works with the functioning of what already exists. Science can try to explain how things worked from the Big Bang, but not why matter exists. That kind of question is for the philosophers, who study the meta-physical (beyond physics).

Some attempts to explain the cause by resorting to some physical concepts such as quantum fluctuations of vacuum confuse the vacuum with nothing: the quantum vacuum exists, it has space and some energy. The concept of nothing, meaning absence of all existence, including space, is not scientific, it is metaphysical. Into nothing, anything can not exist and fluctuate. Other theories talk of multi-universes but by definition are impossible to verify (if we could in some way observe other universes, then they would be part of ours, because our universe is all matter that is within reach in some way). For that reason, those theories are not really scientific.

Let's return to science. At the initial instant, all matter and energy was infinitely small and dense. The Big Bang was the explosion of space at the beginning of time, and from that moment, the matter became operational, with laws that were written in it, and that led the universe to the current state.



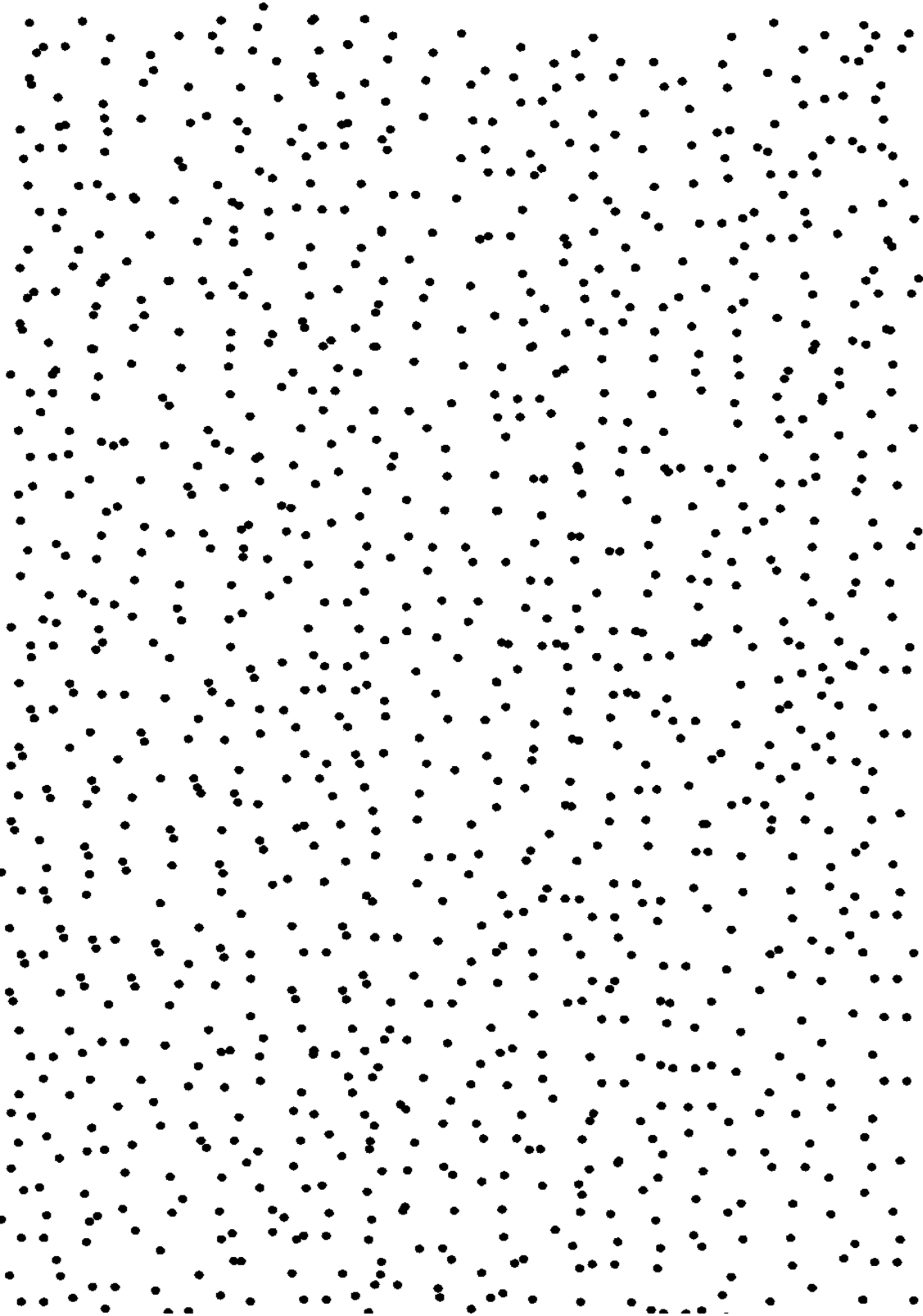


Fig. 10: Photocopy this page on a transparency and then another enlarged 105%.

## Activity 6: There is no center of expansion

On the next page is a drawing (figure 10) with many points that simulate galaxies at a given time. First make a copy on transparent paper and then another on a different transparent paper, slightly expanded (e.g. 105%).

If superimposed on an overhead projector (figure 11a), we get an image that represents the expansion of space over time: match the images at one point, and you can observe the displacement of all radial points very well, which is greater the farther you are from the coincident point. It seems as if the points move away faster the farther they are from the coincident point.

But if the matching is at another point (figure 11b), it is the same. So it is in space: from our galaxy we see that all move away from us, and they move faster the farther away they are from the observer. We think we are in the center of the universe, but we're not, as an observer in another galaxy would see the same thing and that would seem to be in the center. There really is no center.

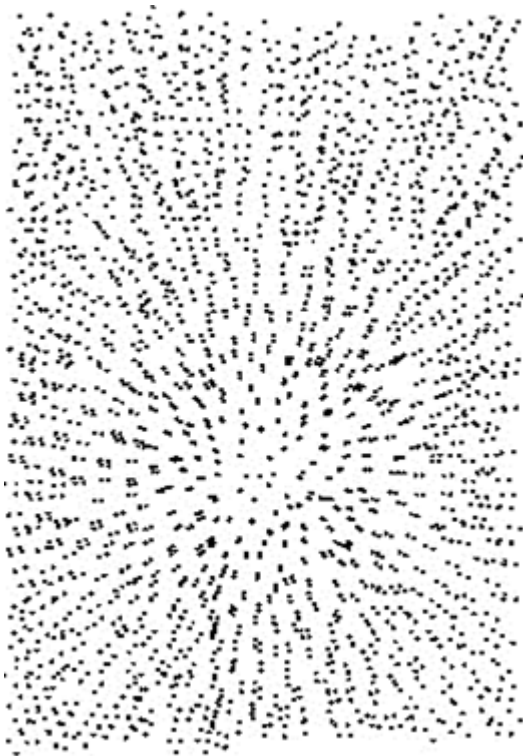


Fig. 11a: Superposition of two slides, one enlarged by 105%.

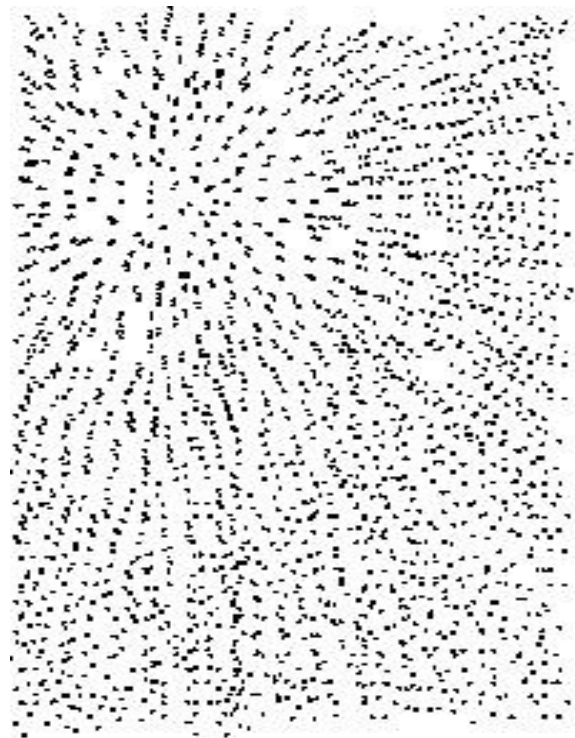


Fig. 11b: To an observer in another point, it also seems that everything moves away of him: there is no a center of the universe.

## Development of the Universe

To get an idea of the later history of the universe, assume that all the time since the Big Bang is compressed into one year from January 1 to December 31 (see figure 12).

In April our Milky Way was formed. In August the sun formed, and the Earth was spherical by end of the month. But it is not until October that oxygen is present in our atmosphere. Although very simple living cells appear on Earth immediately, nucleated cells appear on December 2, and on Dec 12<sup>th</sup> the first multicellular organisms are present. On the 19<sup>th</sup> the first fish appear, as do the plants, insects and amphibians on the 21<sup>st</sup> through the 22<sup>nd</sup>. On the 25<sup>th</sup> dinosaurs appear, lasting until the 28<sup>th</sup>. On the 30<sup>th</sup> the mammals are living on Earth, but it's not until December 31<sup>st</sup>, at 11 pm, when man appears. At 11:57 pm is when Neanderthal man lived, and the painting of the caves of Altamira happened in the last minute. Five seconds before twelve o'clock at night is when Jesus Christ was born. The last century is in the last two tenths of a second.

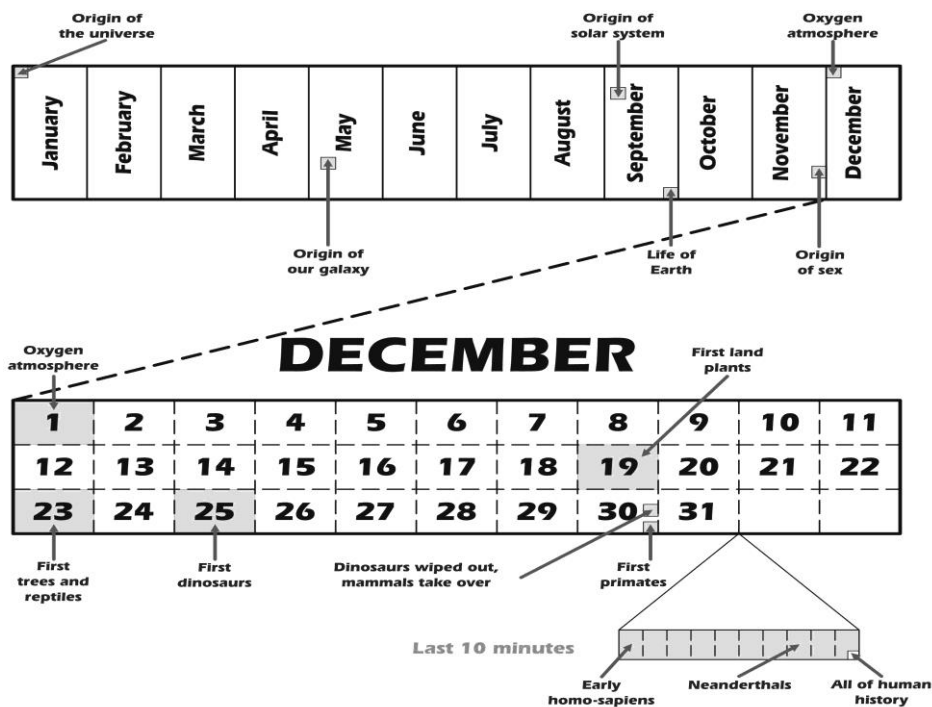


Fig. 12.: The Cosmic Calendar. The history of the Universe compressed to one year. All of recorded history (human civilization) occurs in last 21 seconds

## Microwave background radiation

In the beginning, at very high temperatures, the four forces we now know were unified. Gravity, electromagnetic force, strong and weak nuclear forces (the latter two only act within atoms) were united. Then they separated and formed photons, electrons, protons and other elementary particles. While the universe is expanding, it is getting cooler. After 300,000 years, the temperature dropped enough to allow atom formation, mostly hydrogen and helium. The density decreased, and the photons were free to move in all directions: there was light. Scientists say that the universe became transparent. These photons are now traveling through space, although it has cooled, so the wavelength has increased dramatically (figure 13), and they become much colder photons, which transmit an energy of only 2.7 degrees Kelvin. This is called Cosmic Microwave Background or CMB.

This background radiation was first detected in 1964 by Penzias and Wilson in the United States. They were trying to remove all the noise in their radio telescope when they caught a

7.35 cm wavelength emission which was always present, regardless of where the huge antenna pointed. They reviewed all the installation and even thought that some birds that nested in the antenna could be the cause, but they could not eliminate this background noise. They concluded that it came from a body transmitter, which had a temperature of 2.7 Kelvin – the current temperature of the universe – and was not in any particular place. It was the universe itself that this background radiation emitted, a relic of the Big Bang. Anyone can detect it with an analog TV tuned to a free channel: about one in every ten points you see on screen comes from that background radiation. Those emissions are in the field of microwaves, similar to home ovens, but with very little energy: it could only heat food 2.7 K.



Fig. 13: Over time as space expands, the photons expand in wavelength. This is the microwave background radiation.

Although this radiation appeared remarkably uniform, G. Smoot, R. Mather and his colleagues were able to see very slight variations in measurements made by the COBE satellite (figure 14a), to the order of millionths of a degree. Simultaneously these fluctuations were detected in the ground in the experiment of Tenerife in the Canary Islands Institute of Astrophysics. And in 2001 NASA launched the WMAP telescope to study the background radiation with considerably more resolution (figure 14b).

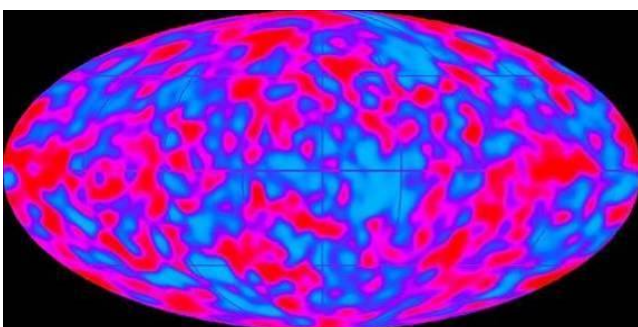


Fig. 14a: COBE image.

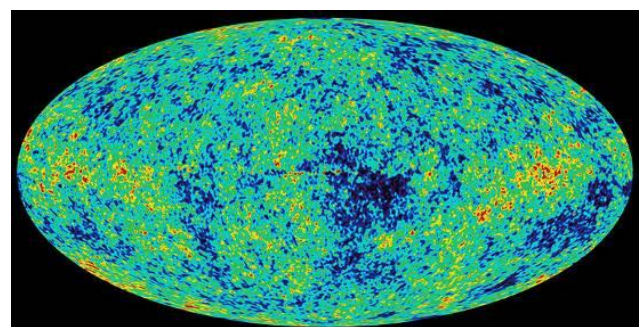


Fig. 14b: WMAP image.

Although small, these variations are the imprints of lumps of matter from which galaxies began to form. We do not know what had caused these fluctuations in density. What we can say is that the “wrinkles” in this area occurred, and condensation began to occur in the proto-

galaxies only a few hundred million years after the Big Bang. Almost simultaneously the first stars had formed in these early galaxies.

### Activity 7: Detection of microwave background radiation

Around 300,000 years after the Big Bang, photons were separated from matter and began to travel freely through the universe. When the space expanded, these photons were extending their wavelength. Now we estimate they have a wavelength of about 2 mm wavelength, which corresponds to the microwave region, and is equivalent to that emitted by a black body that was at 2.7 degrees Kelvin.

Penzias and Wilson, in 1964 first detected the microwave background radiation, a relic radiation that comes very evenly from all directions. The COBE satellite (figure 14a) and later the WMAP (figure 14b) made a very accurate measurement of this radiation in all directions, detecting tiny variations from one area to another, corresponding to what were then clusters of galaxies.

Also we can detect this background radiation with a simple television (figure 15). To do this, tune the TV to an analog empty channel. The image is composed of a multitude of constantly changing points. Approximately 10%, ie one in ten come from the background radiation of the universe.



Fig 15: Some of the points of an analogue untuned television screen comes from microwave background.

## Why is it dark at night?

This was the title of an interesting article that the German Heinrich Olbers released in 1823. Previously, in 1610, Kepler had considered it as a evidence that the universe could not be infinite. Edmund Halley, a century later, noticed some particularly bright areas in the sky and suggested that the sky is not uniformly bright during the night because, although the universe is infinite, the stars are not evenly distributed. Even the writer Edgar Allan Poe (1809-49), wrote on the subject<sup>1</sup>. However, the issue went down in history as the Olbers's Paradox.

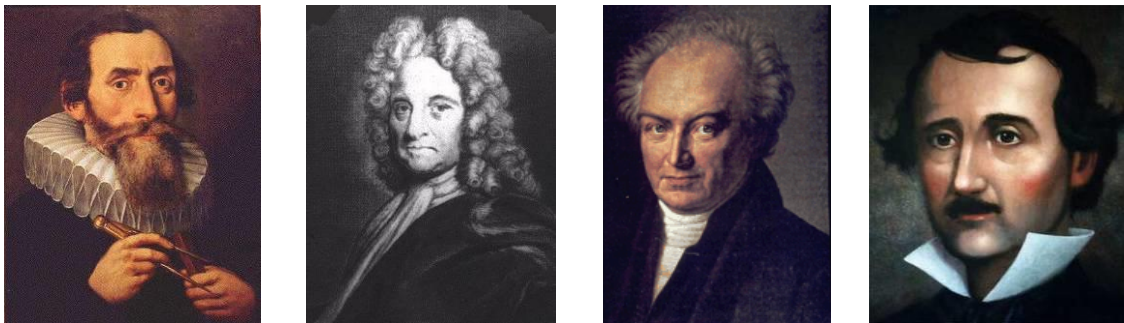


Fig. 16a Johannes Kepler, Fig. 16b Edmund Halley, Fig. 16c Heinrich Olbers and Fig. 16d Edgar Allan Poe

The answer seems trivial, but not so after reading the article from Olbers. Olbers's reasoning led to the paradox that the night sky should be as bright as the most glorious day. Let's see the reasoning.

Olbers's reasoning was based on the following principles:

- 1.- The universe is infinite in extent.
- 2.- The stars are distributed more or less evenly throughout the Universe.
- 3.- All the stars have a similar average brightness across the universe.

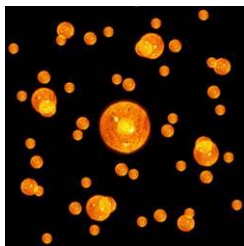
Look at the universe from Earth. Suppose a first spherical shell of stars in the sky at a distance  $R_1$ . The number of stars it contains will be  $N_1$ . Suppose a second spherical shell at a greater distance  $R_2$ . Each of its stars will be illuminated by far less, yet the layer is larger and contains more stars, according to principle No. 2, and counteracts the lesser light (the light intensity decreases proportionally to  $1/R^2$ , and the area of the layer, and therefore the number of stars increases as  $R^2$ ). The conclusion is that the second layer illuminates the Earth just like the first. And according to principle No. 1, there are infinite layers, so the conclusion is that the sky should appear bright at night.

Another way of putting it: if we observe the night sky, where there are countless stars, our eye should always be seeing the surface of a star, and therefore we should see a bright spot there. And if that happens across the sky, it should appear totally brilliant.

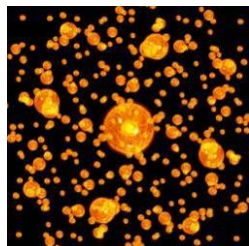
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<sup>1</sup> In "Eureka", a scientific essay published in February 1848, he gave the following explanation for the "empty" dark between the observed stars: "We could comprehend the voids which our telescopes find in innumerable directions assuming that the distance from the invisible bottom is so immense that no ray of light from there has yet been able to catch us".

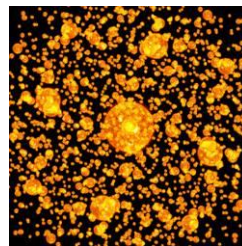
Obviously this is not true. This paradox of Olbers caused a lot of controversy and could not be resolved properly until the early twentieth century with the Big Bang theory. The argument itself is correct, but it fails in its principles. Indeed, with the expansion of the universe, the light from distant stars are at a larger redshift then the further away they are. That implies a weakening in the intensity of radiation, so the principle No. 3 is not correct. We also know that the farther away the star, the longer ago the light left it, so we see it as it was long ago. The more distant stars were formed shortly after the Big Bang, but we can't observe more than that because there aren't infinite layers of stars — the principle No. 1 is also false.



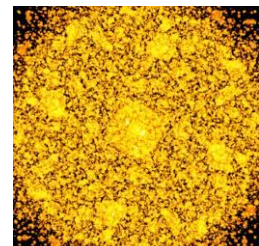
The light coming from nearby stars



But there are also further stars who send us their light



The further, more stars



From any point in the sky we should reach the light of a star

Fig. 17: By wikimedia commons

In the twentieth century, the solution to Olbers's Paradox was resolved with the understanding of the expansion and particularly with the age of the universe, which is not infinite. Fortunately, the night could still be dark!

## Gravitational lenses

Light always follows the shortest possible path between two points. But if a mass is present, then space is curved and the shortest possible path is a curve as seen in figure 18a. This idea is not difficult for students. We can easily show it on a terrestrial globe (figure 18c). Obviously they can understand that on the surface of the Earth the distance between two points always follows a curve.



Fig. 18a and 18b: If the space is curved, the shortest path between two points is a curve.



Fig. 18c: The shortest path above the terrestrial surface is not a straight line.

In general, we can imagine a gravitational lens as an ordinary lens, but in which the deflection of light is produced by a large mass that is in the path of the light, called deflector (figure 19a).

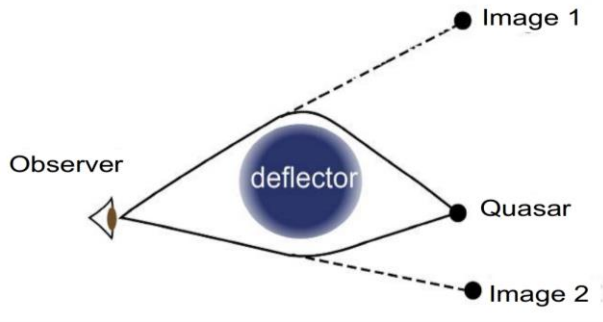


Fig. 19a: The observer sees two images, because it appears as though the light is coming from two different places.

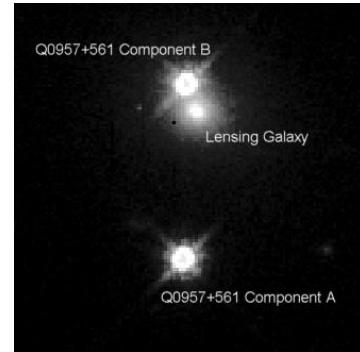


Fig. 19b: Picture of the double quasar Q0957+561 image. The deflector is the galaxy close to the B component.

Gravitational lenses produce a curvature in the beams of light that are emitted by astronomical objects. If these objects are point sources (stars or quasars), they appear to be in a different place from where they actually are, or sometimes even multiple images of the object are produced (figure 19b). If the emitting objects are extended (e.g., galaxies), the images appear distorted as bright arcs (figures 20a, 20b and 20c).

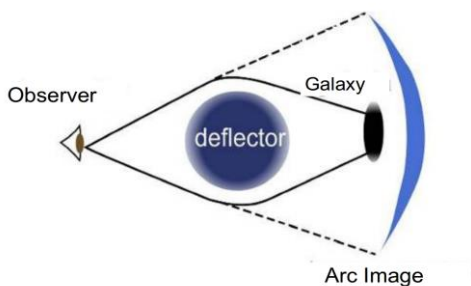


Fig. 20a: If the body diverted is an extended object, the images obtained are a set of bright arcs or a complete ring.

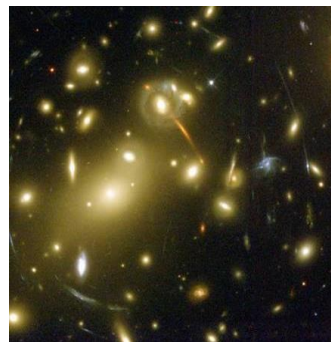


Fig. 20b: Giant luminous arcs formed by the galaxy cluster Abell 2218



Fig. 20c: Complete ring of a galaxy behind the deflector.

## Activity 8: Simulation of gravitational lens with a glass of wine.

We can simulate a gravitational lens using a glass of wine. This experiment allows you to "show" how matter can introduce distortions in the images observed. Now let's simulate the Einstein ring or multiple images. Take a flashlight, place it on the other side of a glass full of red wine or juice and observe the ray of light passing through it.



It is easy to see that this simulation leads to the "distortion of space" that is observed. Simply place the glass on graph paper and look through the white wine (or apple juice). We see the distortion of the graph lines (figures 21a and 21b).

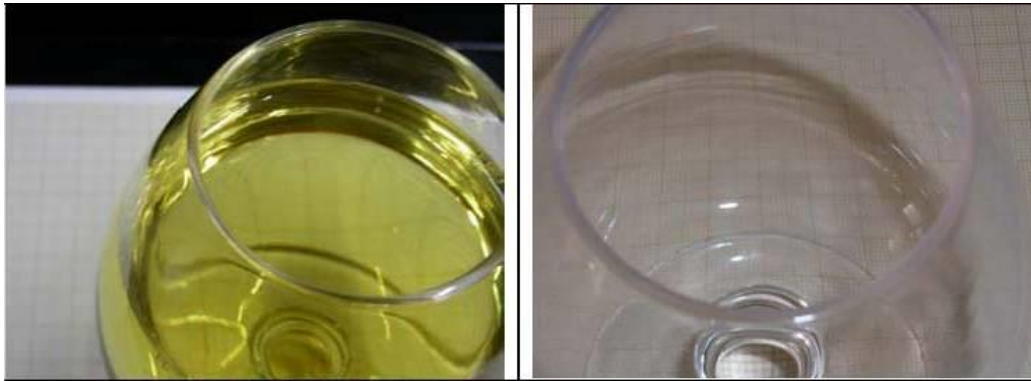


Fig. 21a and 21b: We only can see the distortion of the graph paper if the glass is full.

Looking at the ray of light, we move it from right to left and from top to bottom. We note that the light is not a point: the wine produces images repeatedly and in some cases some arches. This is a consequence of the glass acting as a lens that distorts the light trajectory. In particular, we can sometimes see an amorphous figure, or a bright red dot, four red dots or a red bow between points (figures 22a, 22b and 22c).

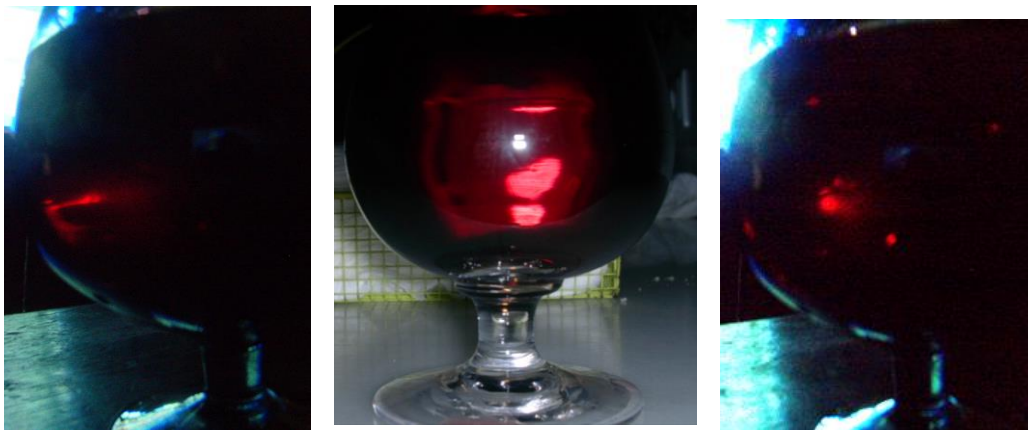


Fig. 22a: The flashlight beam is distorted as an arc between two bright red spots, Fig. 22b: like an amorphous rectangle, and Fig. 22c: the Einstein cross.

We can also simulate the gravitational lens looking through the glass foot of the wine glass. If we put the foot of the glass on a graph paper and look through it, we can see the deformation of the grid (figure 23).

Moving the foot of the glass slowly from right to left above an object, (e.g., a red circle about 3 cm), we can reproduce the shapes observed through gravitational lenses (figures 24a, 24b and 24c).



Fig. 23: Grid deformation.



Fig. 24a, Fig. 24b and Fig. 24c: The glass foot can simulate various shapes made by gravitational lenses: arc segments, images of points, and Einstein rings.

## Bibliography

- Moreno, R. *Experimentos para todas las edades*. Ed. Rialp., Madrid. 2008.
- Moreno, R. *Taller de Astrofísica*. Cuadernos ApEA. Antares, Barcelona. 2007.
- Moreno, R. *Historia Breve del Universo*. Ed. Rialp., Madrid. 1998.
- Moreno, A, Moreno, R. *Taller de Astronomía*. Ediciones AKAL, Madrid. 1996.
- Rianza, E, Moreno, R. *Historia del comienzo: George Lemaître, padre del Big Bang*. Ediciones Encuentro, Madrid, 2010.
- Ros, R.M, *Experiments and exercises involving gravitational lenses*, Proceedings 1st ESO-EAAE Astronomy Summer School, Barcelona, 2007.
- Ros, R.M, *Gravitational lenses in th classroom*, Physics Education, 43, 5, 506, 514, Oxford, 2008.

## Internet Sources

- <http://www.dsi.uni-stuttgart.de>
- <http://georgeslemaitre.blogspot.com/>
- <http://www-ra.phys.utas.edu.au/~jlovell/simlens>
- <http://leo.astronomy.cz/grlens/grl0.html>